Coexistence of Physical and Crypto Assets in a Stochastic Endogenous Growth Model

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Abstract

We study a stochastic dynamic model with risky real investment and a positive long-term growth rate. With growing wealth, the economy gets clogged with increasing complexity costs (the classical “Leviathanian” inefficiencies in the form of implicit taxation and abuse of power, red tape, outlays on conflict resolution between special interest groups, etc.). To escape the Leviathan, agents can, in addition to the usual investment in physical capital, access the universe of crypto assets outside the reach of the mainstream state-supported economy. Crypto assets enjoy no legal protection, so converting them back into the real life consumption good is risky (due to digital criminality, hacking, regulatory crackdowns, etc.). A global ergodic solution is found for this model, demonstrating that crypto and conventional assets are capable of long-term coexistence, although the use of crypto assets, far from being universal, tends to be the choice of the wealthier part of the population.

Key Words: crypto assets, investment, DSGE, full distribution solution, ergodic distribution

JEL Codes: E02; G23; C61; C63; D58; E26

1 Introduction

Observers of the rise of Bitcoin and subsequently other crypto-currencies and assets during the last decade must have noticed that their users are largely recruited from those who are not simply looking for investment diversification, but are, more generally, dissatisfied with the conventional institutions that are supposed to provide support for their savings and wealth creation. Examples range from retirees whose pension savings are being eroded by monetary easing, through overregulated medium-sized enterprises (both in developed countries) to high-net-worth individuals and families living under politically oppressive or unstable regimes in emerging economies.

Not all, rather only a small part of, institutional complexity costs have the nature of taxes. Instead, among the manifestations of the costly complexity of wealth management is the persistence of negative (effective) interest rates on a wide range of conventional deposits. Negative rates are becoming policy-engrained, while policymakers now seem more averse to recessions than their official mandate would require, compared to a couple of decades ago. Savers are then forced to react to the financial repression by seeking non-conventional (and usually illiquid) store-of-value alternatives. Another novel sort of complexity facing private individuals is related to the mushrooming administrative obstacles in advanced countries (accompanied by supportive rhetoric of the official punditry) to the use of cash. For those who value privacy (there are substantial numbers of such beyond those involved in illegal activities), this means additional outlays on transaction anonymization.

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In general, as modern societies tend to become more complex and difficult to operate (Turchin et al., 2018 and Whitehouse, 2019; see also Tainter, 1990, for a historian's perspective), many institutions, some dating back to the Industrial Revolution of the 19th century or even older times, and which we mostly take for granted, have a hard time playing their designated roles. Certain ones only survive in the form of empty shells void of actual function. The use of others according to purpose consumes much more resources today than it did when they were created. It is enough to look at the largely self-serving judicial system, whose operators collectively benefit from the legal process (in particular, but not exclusively, in the area of civil law) becoming increasingly expensive, unwieldy, and clogged. One of the consequences is that the enforcement of many laws becomes fairly improbable for those unwilling or unable to spend extravagant sums to pay their way through the courts in a procedure that is at risk of not being completed within one's lifetime. State bureaucracy, to which innumerous rival lobbies and NGOs are connected with the umbilical cord of the "revolving door effect", offers a similar picture. The state powers (to which, in the world of today, one must add other comparably influential organisms such as supranational corporations, e.g. big tech, and other comparatively powerful special interests) were once compared by Thomas Hobbes, one of the leading political philosophers of the Enlightenment, to the Leviathan creature of the Old Testament Book of Job.[1] In our times, one finds that the ageing beast has become much less nimble, but all the more voracious.

To be sure, there also exist numerous manifestations of social complexity not due to self-interested abuse of the tragedy-of-the-commons type. A lot of costly complexity is an unintended consequence of well-meant collective effort to enforce prudent handling of systemically important risks, financial and other. For instance, the various safety nets and prudential regulation measures that were introduced in the aftermath of the GFC may have reduced the vulnerability of the global financial system to a quick spread of market turbulence. However, this has come at a cost of rapidly expanding financial supervision and reporting mechanisms. The need to comply with them is becoming increasingly resource-consuming not just for the providers, but also for the users of financial intermediation. So, while we have been able to reduce the threat of an uncertain breakdown, the growing precautionary effort has become a certainty for the indefinite future. Finally, one can be absolutely certain that the newly discovered epidemiological and other health risks, once the COVID 19-pandemic has made them a non plus ultra political and social concern, are currently creating a separate dimension of societal complexity phenomena.

It is important to note that the aforementioned problems are naturally felt more intensely as individual and aggregate wealth grows. The marginal benefit of every extra dollar earned within mainstream society falls, since any nominal wealth increase becomes costlier to secure, both with time and in relation to the wealth level. Under these circumstances, the crypto asset ecosystem, which (seemingly) knows no state borders, red tape, or special interests able to bend the rules, offers an attractive escape route. There, transparency and predictability if not of the counterparties, then at least of the procedure is the central promise.[2]

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[1] According to experts in the subject, this metaphor is a natural reflection of Hobbes's understanding of political force, since both "Leviathan and Hobbes's sovereign are unities compacted out of separate individuals; they are omnipotent; they cannot be destroyed or divided; they inspire fear in men; they do not make pacts with men; theirs is the dominion of power." (Mintz, 1989, p. 5)

[2] In principle, any asset able to offer a way out of deterministic complexity at the cost of random losses would perform the same function. However, in the modern age, once one requires such an asset to be truly global, universally accessible, and free of centralized regulation, the set of possibilities is quickly reduced to the ones offered by existing and conceivable crypto assets residing in the cyberspace. There is simply no other way to reach out to all the interested parties than by creating digital commodities and distributing them online. This is, essentially, all we require from the crypto asset in this paper.

As is well known, the viability of crypto assets stands and falls with their take-up numbers, so they are only worth anything as long as enough people are willing to hold them. In principle, a major fall in trust can reduce any crypto wealth to zero. This is why there is a popular perception of crypto assets as bubbles *par excellence*: unproductive assets with positive prices enabled by purely psychological factors. Still, if the only alternative to crypto assets is the constantly falling marginal benefit from conventional assets, then at some point a sufficiently wealthy agent has nothing much to lose by withdrawing into the crypto world for store-of-value reasons. Although the Leviathan does not protect his rights there and is unable to enforce contracts, whereas breakdowns, trust abuse, and fraud are rampant in the cyberspace, these are mere possibilities, whereas the continuously flattening absorption curve that is typical of effective earnings on conventional assets is a near certainty. Many decide to take their chances. Crypto assets start resembling a rational bubble (cf. Martin and Ventura, 2018), although in our set-up, as opposed to other rational bubble models, we do not need to posit the existence of an exogenous variable responsible for its positive market price: there is a generic endogenous crypto demand.

The question arises whether the crypto option is merely an ephemeral appearance created by special circumstances of modern society and bound to disappear as soon as its temporary benefits are wiped out by an episode of major turbulence followed by a crisis of confidence and an ensuing terminal fire sale (i.e., a classical bursting bubble), or whether its existence is a part of some emerging sustainable economic order. Can this order be characterized in abstract terms as an equilibrium with intuitive choice-theoretic foundations? The model proposed in the present paper tries to answer this question in a relatively simple producer-consumer environment with risky investment opportunities. An important feature of this environment is a persistent income inequality in a large agent population. Then, it turns out that the bubbly aspects of crypto, although present, affect different agent cohorts to a different degree. Fire sales are widespread among the poor, who are prone to giving up on crypto in distress. On the contrary, middle- and high-income agents generate a stable crypto demand that improves aggregate (conventional) investment and consumption levels.

Formally, we set up a dynamic stochastic discrete time optimization model with two sources of uncertainty: TFP and crypto-to-fiat conversion loss. The market for crypto assets clears in every period. Differently from many other DSGE applications, our solution method does not rely on perturbation analysis, but instead belongs to the full distribution category. Agents have Markov policy functions of two state variables (physical capital and crypto holdings) that define optimal consumption, acquisition of new tokens, and back-conversion of previously held tokens. The solution method is one of the central elements of the study.

We are able to compute the equilibrium for a broad class of initial capital and crypto asset holding distributions. In addition, we provide a numerical procedure for deriving an approximation for an ergodic distribution of the two asset holdings across the agent population, i.e., such that is preserved by the model dynamic under optimal policies.[3]

Naturally, it would also be possible to construct an additional asset class with features imitating existing crypto, held in equilibrium in a positive amount alongside conventional assets, in a considerably simpler static model. However, our objective in this exercise is to experiment with the infusion of crypto assets into a much richer stochastic growth environment and see how they fare under conventional macroeconomic dynamics. Ultimately, the results should pave the way for introducing crypto into the existing macro DSGE model class. To assure that this introduction happens

[3] Since we consider a balanced growth model, ergodicity here refers to the distribution of an appropriately discounted (effective) vector of state variables. The exact specification is given in section 3.

on a solid formal basis, we do not just state the model, but also compute a full distribution solution in all the cases considered. The numeric solution techniques developed for dynamic stochastic optimization problems in discrete time allow one to address the cases of both idiosyncratic and aggregate shocks. However, the present exercise concentrates on the idiosyncratic shock case, leaving the issue of the crypto economy response to a systemic disturbance to subsequent research.

We find, among other things, that

(1) for any non-degenerate initial distribution of physical capital and crypto asset holdings that result in positive aggregate output growth there exist optimal investment and crypto transaction policies such that agents with capital endowments above some minimal level hold non-negative crypto quantities; the market-clearing crypto price in this equilibrium is positive;
(2) the aggregate growth rate of physical capital/output is higher when crypto assets are allowed, as opposed to the same economy without the crypto option, and the same is true for aggregate consumption;
(3) when the crypto opportunity exists, it is always used, even by a subset of agents with zero initial crypto endowment, provided their physical capital endowment lies above a certain threshold;
(4) under the ergodic distribution of physical capital/crypto quantity pairs, physical capital levels are positively correlated with crypto holdings;
(5) under the ergodic distribution, agents with higher crypto holdings invest somewhat less in physical capital than those with lower crypto holdings; nevertheless, the aggregate investment is higher than in the same economy without crypto and under physical capital distributed ergodically, since the agents are wealthier on average (support of the marginal physical capital density of the ergodic measure of asset pairs is positioned to the right of the ergodic physical capital distribution support in the no-crypto economy).

While the ergodic asset distribution is the natural way to describe the hypothetical long-term condition of the dynamic system defined by the model, optimal policies exist for every initial asset distribution. We do not directly address the question of convergence speed (or its exact meaning, given that ergodicity refers to discounted, or growth-corrected, and not original, quantities), but observe that, qualitatively, the computed investment policy functions under ergodic and non-ergodic distributions are quite similar. For instance, they always generate a region of crypto non-adoption (in which originally poor agents reside) as well as another region in which agents do not consume anything out of conventional income. Specifically, somewhat more well-off agents than those who pass on crypto completely, split their conventional earnings between new investment and new token purchase, whereas consumption is fully financed by token back-conversion. In this way, they avoid paying any tribute to the Leviathan and remain only on the production side of the conventional economy (they have to make some real investments, since crypto assets do not have direct productive uses). On the contrary, wealthy agents seem not to mind a limited Leviathan presence in their lives, but improve their welfare by engaging heavily with crypto to the detriment of physical investment (in comparison with the no-crypto benchmark). In any case, their consumption out of conventional income is lower than in the benchmark.

The aforementioned findings suggest that crypto assets, notwithstanding their partially bubbly nature, are able to improve welfare in situations in which they help reduce structural deficiencies in an economy. Still, the derivation of equilibria in the present setting hinges on the presence of multiple agents with sufficiently diverse asset endowments, i.e., it would be impossible in a classical representative agent model. To be precise, a representative agent in our setting can only exist initially, since unevenly distributed exogenous wealth shocks (in the form of random total factor productivity and random crypto conversion losses) in all subsequent periods lead to a permanently heterogeneous agent population.

1.1 Literature review

Since even the earliest crypto assets are barely ten years old, there has not been enough time for an extended theoretical literature on them to emerge. Nonetheless, given the early-recognized high level of policy relevance, academic journals soon began to publish policy-oriented essays that strived to pin down the proper conceptual role of cryptocurrencies as an object of economic inquiry (Dwyer, 2015, Yermack, 2015, Weber, 2016). At the same time, taking the self-proclaimed aspirations of Bitcoin and its imitators to create a world without fiat money all too uncritically, the bulk of theoretical work on the emerging crypto seeks inspiration in the existing currency competition literature. So, Fernández-Villaverde and Sanchez (2016) analyze competition among privately issued digital currencies (tokens) and the monetary authority issuing fiat money, in a dynamic optimization set-up with production and decentralized markets with random matching. Their point is to demonstrate the inefficiency of private moneys, but with the ability of the fiat money issuer to implement an efficient policy under the pressure of private competition. The topic is further developed in Fernández-Villaverde (2018). Cryptocurrencies as "outside money" in the paradigm of Kareken and Wallace (1981) are examined in Garratt and Wallace (2018) to demonstrate the possibility of their exchange rate indeterminacy. Demand for alternative means of exchange, including one that shares with cryptocurrencies (not just Bitcoin) a purely private supply and operation outside of any official policy perimeter, is modeled in Schilling and Uhlig (2019a,b) in a simple endowment economy environment. A tentative exploration of a virtual currency exchange rate can be found in Bolt and van Oordt (2019), in which a two-period model of currency choice under specific assumptions about the new currency acceptance drivers (technological adaptability among them) is used to state sufficient currency success conditions. Brunnermeier and Niepelt (2019) provide a model in which allocational equivalence of a private means of exchange (encompassing some, but not all existing types of cryptocurrencies) with the fiat money can be achieved by a monetary authority under very restrictive, and therefore hardly practicable, conditions. All the mentioned papers view crypto assets strictly from their means of exchange side, largely leaving the store of value role out.

To the extent crypto assets feature bubbly properties, the literature on rational bubbles (Martin and Ventura, 2018) overlaps with the topic of the present paper. However, differently from many existing rational bubble models, our foundational assumptions concerning market psychology are very modest. In our model, once even a few agents get interested in crypto assets, the latter start providing a store-of-value service. Therefore, the bubble side of crypto assets, although present, is subordinate to them being a conduit to the "silent anti-Leviathan insurgency" mentioned earlier.

Empirical crypto analysis is somewhat richer than economic crypto theory, mostly due to the abundance of publicly accessible data on Bitcoin and other cryptocurrency prices, often on transaction volumes as well. This allows one to get deeper insights into individual market segments. So, Kristoufek (2015) uses wavelet analysis not just to identify the main Bitcoin price drivers, but also to separate the special role of Chinese investors in the Bitcoin market. The identified speculative impulses coming from Chinese investors underscore our point made earlier about the significance of social climate for the decision to engage in crypto investments. Early studies of the bubble properties of Bitcoin can be found in Cheah and Fry (2015) and Cheung et al. (2015), i. a. There are also studies that use a rational bubble-inspired OLG-model framework that informs an empirical estimation of the fundamental cryptocurrency price and sources of deviation therefrom (e.g. Biais et al., 2018). Some research is emerging regarding the empirics of initial coin offerings (Rhue, 2018, Burns and Moro, 2018). As far as we are aware, there have not yet appeared any attempts in the academic literature to link crypto demand to wealth, social status, or other sociological categories.

The rest of the paper is organized as follows. Section 2 describes the model and formulates the generic optimization problem. Section 3 states the optimality conditions and introduces the solution method. Section 4 explains the equilibrium concept used and outlines the equilibrium search procedure. It also describes the numerical results obtained for both the bivariate lognormal and ergodic asset distribution cases. Section 5 concludes.

2 Model

2.1 Production

We consider an infinite-horizon discrete-time closed yeoman economy populated by a large number of infinitely lived small producers-consumers. Their distribution is described by a measure μ on the measurable "agent space" Ω to be specified later. Each agent has a standard power production function[4] concave in own physical capital k:

$$y = Af(\overline{K}, k) = A\overline{K}^{1-\alpha}k^{\alpha} \tag{1}$$

(time subscripts are omitted for simplicity, as all values refer to period t). Here, α is the own capital share in output y, A is stochastic total factor productivity (TFP), and $\overline{K}$ is the aggregate (or average per-capita, given that the mass of agents is normalized to unity) physical capital level:

$$\overline{K} = \int_{\Omega} k(i)\mu(di). \tag{2}$$

This means that we employ the learning-by-doing externality of the Harrod (1939), Domar (1946), Romer (1986), and Lucas (1988) endogenous AK-growth type. Physical capital partially depreciates every period:

$$k_t = (1-\delta)k_{t-1} + I_t, \tag{3}$$

with I_t being new investment in period t. Productivity shocks across periods and across agents are i.i.d.

Investment in physical capital generates output next period that can be spent on consumption, c, new physical capital (at constant unit price), I, or conversion to crypto assets at the prevailing market price. (All agents are price takers.) The unit of the consumption good is the numéraire.

2.2 Tokens

We imagine crypto assets to be available through a virtual investment fund, so the price means the cost of one share of that fund. This abstraction is meant as a shortcut for a variety of really existing possibilities to acquire crypto assets (mining, participation in an ICO, buying at a crypto exchange, etc.) that we do not model explicitly. Accordingly, underlying the purchase fiction of one crypto fund share in our sense, both a purchase in a secondary market and mining (in the latter case, one actually purchases computer time, energy, overheads, etc. instead of the token itself) can take place. Once acquired, the share can be distributed – presumably optimally – among the available tokens in a further unspecified way. For time- and space-saving reasons, we will use the term "token" instead of "crypto fund share" throughout the text unless the term causes confusion.

[4] Numerical exercises show that the quantitative results of the model may be sensitive to the production function specification. Therefore, the current standard Cobb-Douglas choice should only be viewed as an experimental first pass.

To accommodate a non-trivial market for tokens in the model, we resort to another fiction that constitutes a deviation from the representative agent baseline. Namely, we think of the agent as a household of two in which one member is responsible for buying new tokens and the other for reconverting currently held ones into conventional goods. The two members do not coordinate their actions within the period, taking the choice of the other as given. A more widespread artificial device making it possible to create a non-trivial transaction volume in an asset market would be to exogenously impose agent population heterogeneity in terms of preferences or technology. We prefer the variant with the multimember household for a number of reasons. First, it better responds to our intention to analyze a set of agents with identical structural parameters, whose members only differ in endowments. Second, in the case of crypto assets, acquisition and sale are not always symmetric activities (only transactions on a crypto exchange are); mining and ICO participation are one-way only, so it is not unnatural to separate the buyer and seller roles. Even with regard to crypto exchanges, their users, outside of the narrow group of professional traders (and those in no way exhaust our idea of the fictitious yeoman household in this model), approach them with a qualitatively different perspective depending on the direction of their trade: buyers seek relief from the Leviathan pressure in the future, whereas sellers cover current consumption needs.

Another minor asymmetry between crypto buyers and sellers is in the information imperfection faced by the seller. We assume that at the moment of sale quantity choice, the agent does not know the realized value of the conversion shock. The assumption is not central to the results, but appears to generate smoother and neater numerical outcomes. Besides that, the value of the back-conversion proceeds in terms of the numéraire is obtained by multiplying the conversion function (see below) by the market price. Accordingly, the seller cannot separate price and conversion loss risks, which explains why it would be unnatural for the agent to condition on price when selecting optimal policies (see section 3). Note that for the buying member's decisions, the value of the conversion shock is unimportant, since he takes sales proceeds as an exogenous lump-sum and does not examine their composition.

If H is the amount spent by the crypto-purchasing member of the agent household to buy tokens at the start of the current period and p is the current price, H/p is the number of new tokens. During the same period, the second household member decides on how many of the existing tokens, S, will be converted back into conventional consumption units. At the end of the period, the joint actions of the two household members result in the following evolution of the numbers of crypto fund shares in the agent's possession:

$$x_t = \left(1 + R(\overline{X_{t-1}})\right)x_{t-1} - S_t + \frac{H_t}{p_t}, \tag{4}$$

where R is the crypto-internal rate of growth (the source of which may be an ICO, mining, and whatever creation mechanism the issuer chooses) dependent on the aggregate number of tokens in existence in the previous period: $R_t = R(\overline{X_{t-1}})$.

Analogously to (2),

$$\bar{X}_t = \int_{\Omega} x_t(i)\mu(di) \tag{5}$$

for every period t. We allow for a network externality making the rate of growth of crypto wealth lowest (nominally zero, but effectively negative due to the random loss rate; see below) when no one is interested in it, and strictly growing with the number of tokens in circulation. In other words, the crypto universe only exists as long as people care about it. At the other extreme, function R is assumed to have a finite positive limit at infinity, so the externality becomes irrelevant as soon as crypto assets take off and become part of a balanced-growth equilibrium.

The reason people would like the crypto universe to persist is that they can (hope to) convert crypto back into fiat money: as long as there are willing crypto buyers, one can always sell, even though the conversion costs are uncertain. These costs include not just the price at a crypto exchange which equalizes aggregate supply and demand, but also cyberspace-specific risks. Namely, the crypto universe is a world without legal protection or security of contract enforcement. There is poorly detectable and nearly unprosecuted fraud, hacker attacks, administrative intrusions by state authorities, technical incidents, etc. We summarize these effects by means of a stochastic conversion loss rate, ranging from zero in the most favorable case (although with low probability) to an upper bound able to deprive one of more than two thirds of the crypto asset value. The latter extreme is also a low-probability event; losses of several percentage points are most common. The number of consumption units available after the back-conversion of S tokens is equal to $pL(l,S)$, with p being the current token price and function L depending on the random agent-specific loss parameter, l, and the amount of tokens, S (the loss risk is parameterized so that L is increasing in l; the latter is a strictly positive random variable with support bounded by unity). L is strictly increasing and strictly convex in S on the positive half-axis, $L(l,S)<lS$ and $L(l,0)=0$ for every l. One can think of $L(l,S)$ for every fixed l as a function that takes off at the origin and then asymptotically approaches a straight line with slope l but lies below the diagonal. This means that the marginal revenue from back-conversion is dominated by the below-unity L-slope at the intercept for tiny conversion quantities (S near zero), but becomes close to l (which itself can be unity, i.e., no conversion losses, in the most favorable case) for large S. This construction was adopted to reflect the fixed back-conversion costs relevant for very small crypto transactions. A typical conversion curve is depicted in Fig. 1.

For simplicity, we assume i.i.d. conversion shocks both across periods and across agents (although it would be interesting to investigate the effect of a population-wide common shock component to study, for instance, the implications of a systemic event in the crypto market; this is a topic of future research). The support of l is assumed to lie in the interval $[l_{min}, 1]$ with $l_{\min}>0$. In this paper, we work with a discrete distribution with only three realizations, $l_{\min}$, l_{middle}, and 1, which speeds up the calculations without affecting the nature of the results. The same is done for TFP shocks. Repeated random draws from this 3x3 shock table are able to generate a very dense set of asset pair (physical capital and crypto) realizations even after a relatively small number of periods. This set approximates continuum sufficiently well for our purposes (although, in the calculations, we return to a finite, even though big, number of points due to the need for discretization), so small-size event spaces of discrete shocks are enough in the present context.

2.3 Financing consumption

Why would anybody want to hold crypto assets if their back-conversion entails a material loss risk? The answer lies in the nature of conventional goods production in the economy discussed. The societal complexity mentioned in the introduction takes the quantitative form of a decreasing effective rate of output absorption by its producers.

Let $a=y-I$ be the agent's output less savings in the current period. This is the "pre-Leviathanian" disposable income coming from his production facility. Given the losses arising in complex societies for wealth protection reasons (both in the form of taxation and other outlays on handling increasing complexity), the actual amount available for consumption will be $n(a)$. We assume a strictly increasing but concave function n, $n(a)<a$ for all positive a. The wasted difference increases convexly with a. A

typical function n is shown in Fig. 2 (cf. the dotted 45-degree line, of which n falls increasingly short with growing a).

Observe the positive intercept of the curve in Fig. 2, as if there existed some universal basic income (UBI) arrangement for the poorest in the economy. This is done for technical reasons, as infinite marginal absorption for zero-income agents, although not adding anything to the analysis of the model, would result in perverse behavior of the numerical solution in the close vicinity of the origin. In equilibrium, we do not have anyone with zero income, so the UBI assumption is formally moot, but it removes the need to seek interpretation of counter-intuitive endpoint effects. For the same reason, we smooth out and remove the infinite marginal product of capital at the origin of our production function in (1) by means of a special factor only relevant for small k-values (not shown in (1) for simplicity).

Only the funds taken out of the official economy and exiled in the crypto universe are exempt from the complexity burden. (Remember, this was exactly the openly declared objective of the first crypto currency creators.) Therefore, the agent has the opportunity to withdraw a part of produced income from under the Leviathan's muzzle, at the expense of reducing not only consumption out of legitimate sources, but also investment in future productive capital. At the same time, the agent can convert some of his pre-existing crypto holdings into fiat money and add them to his consumption expenditure, again without the Leviathan being able to snatch any of these conversion proceedings for itself. Formally, the overall consumption of the agent is

$$c = n(a - H) + pL(l, S). \tag{6}$$

Remember that $a=y-I$ is the current productive income less savings and H is the amount spent on token purchase. When I and H are chosen optimally, the agent escapes the increasingly inefficient Leviathan into the crypto universe and raises current consumption at the expense of curtailing investment in his conventional production facility. In aggregate, there is also adverse pressure on the growth rate due to the TFP-externality of the posited endogenous growth mechanism. And even this growth-sacrificing "re-routing" can only function if some other agents do the same and keep the crypto universe away from collapse. Can such an economy possess a dynamic equilibrium in which crypto assets survive and make a positive contribution to growth? Unlikely as it may seem in view of what was said above, there are equilibria that support both growth and crypto, and they also turn out to be welfare-superior to economies without the crypto option.

Since this is a growth model, we adjust the otherwise standard power utility in every period to include the reference level of previous period consumption. Thus, let

$$\bar{C}_{t-1} = \int_{\Omega} c(i)_{t-1} \mu(di) \tag{7}$$

be the aggregate (average per-capita) consumption in the preceding period, and work with a time-separable utility that in each period is given by the usual power of the consumption index:

$$u(c_t) = \frac{1}{1-\frac{1}{\sigma}} \left(((\bar{C}_{t-1})^{1-\rho} (c_t)^{\rho}) \right)^{1-\frac{1}{\sigma}}. \tag{8}$$

Here, σ is the intertemporal substitution elasticity (and risk aversion) parameter and ρ is the agent's current consumption share in the Cobb-Douglas consumption index. $\bar{C}$ is treated by the agent parametrically. The index can be also rewritten as

$$\bar{C} \left(\frac{c}{\bar{C}} \right)^{\rho}$$

(the left-out time subscripts are the same as in (8)) and interpreted as the reference consumption level (the last-period average per capita) times the ρ-power of the ratio of own current consumption to the reference level. That is, the agent values his individual consumption improvement compared to the most recent recorded population average.

In the calculations, we take a value of ρ close to unity.

The agent's problem at any given date t is to select the stream of triplets (I,H,S) over all future periods to maximize the intertemporal expected utility

$$U_t = E_t\left[\sum_{\tau=0}^{\infty} \beta^\tau u(c_{t+\tau})\right]$$

(β is the usual discount factor; in growth models it is usually assumed to be small enough to make the discounted sum of the aggregate consumption utilities finite) with u defined in (7) and (8). The optimization is subject to technology constraints (1)–(5), the resource constraint (6) and the non-negativity constraints on token holdings ($x \geq 0$, i.e., crypto cannot be short-sold), legitimate income (a-$H \geq 0$, i.e., one cannot request a "tax credit" from the Leviathan), and consumption, in every period.

The parameterization of the model is summarized in Table 1.

The conditional expectation is taken over the future realizations of both exogenous risk factor processes. We work with discrete distribution examples in the numerical part of this study, for which the following notation will be used. TFP can take a finite set of values $\{A_\kappa\}_{\kappa\in\mathrm{K}}$ with probabilities $\{\vartheta_\kappa\}_{\kappa\in\mathrm{K}}$, whereas the conversion shock takes a finite set of values $\{l_\lambda\}_{\lambda\in\Lambda}$ with probabilities $\{\pi_\lambda\}_{\lambda\in\Lambda}$. As already mentioned, for our purposes in this paper it is enough to work with sets K and Λ consisting of just three elements each.

As mentioned earlier, the current-period realization of the token conversion shock is unknown at the time of that period's decisions, whereas the TFP shock is known (i.e., the agent knows the output level from (1) when deciding upon new investment and operations in the crypto market). We are looking for Markov policy functions, i.e., such decisions that, for every period, only depend on the values of income and asset variables y, k, and x available at the start of the period.

The usefulness of dynamic stochastic models of the type defined above is seriously limited by difficult access to their actual solutions (the popular "non-stochastic steady state" fiction should not count here, since it is largely immaterial to the behavior of the true solution). Therefore, our effort is focused on developing numerical procedures that inform us about the true (also called global, or full-distribution) solution of the generic individual optimization problem. Subsequently, we will derive equilibrium properties consistent with these solutions under appropriate aggregate constraints. The non-stochastic steady state, or perturbation exercises around it, plays no role in our analysis; it could not even if we wanted it to, which we don't. What can remain concealed in classical macro exercises such as the calculation of IRFs for conventional policy shocks, would be absurd, and conspicuously so, in the process of developing explanations for long-term investment patterns. Here, we model attitudes to crypto and some of their macro implications, without pretending to know the future. Our long-term equilibrium will be a time-homogeneous dynamic system in which the state variables follow an ergodic distribution. We will provide a discrete measure that approximates this distribution (one could call it "near-ergodic") for the given discrete approximation of equilibrium optimal policies.

3 Optimal policies

3.1 Preliminaries

As was already said in the previous section, we assume that, at the beginning of every period, there are many agents with many different physical capital and token possessions. Shock realizations transform the population distribution after every period. It is natural to ask what happens with an arbitrary distribution at the end of the period if agents' actions are subjectively optimal, but also, what shape this population is likely to take after many periods. Therefore, we consider two problems: one is the calculation of optimal policies under an arbitrary initial distribution, and the other is the calculation of an ergodic distribution, i.e., such that is invariant under agents' optimal actions given this distribution.

Before proceeding, we observe that population statistics enter the agent's problem only through population averages $\overline{K}, \overline{X}$, and $\overline{C}$, as defined in (2), (5), and (7), as well as through the crypto market clearing condition in every period:

$$\int_{\Omega} H_t(i)\mu_t(di) = p_t \int_{\Omega} S_t(i)\mu_t(di), \tag{9}$$

which pins down the token price p_t. Note that, in general, one must also include the time index of the population measure μ in (2), (5), and (7) in the same way as in (9), unless the ergodic measure is meant. However, given the Markov policy functions that we analyze, only the measure at one particular date (the one with which the agents start the period) enters the calculated solution of the agent's problem, so the time index can usually be omitted without causing ambiguity.

Some features of the optimization problem are the same regardless of the ergodicity condition. We will state them after the introduction of a more convenient set of state variables.

To begin with, in view of the information structure introduced (the TFP realization is known at the start of the period), and provided the next-period physical capital is properly expressed through policy functions, an agent's income less investment in any period t is more parsimoniously expressed by means of the gross (i.e., cum-depreciated capital) disposable income variable

$$q_t = A_t(\overline{K}_t)^{1-\alpha}{k_t}^{\alpha} + (1-\delta)k_t$$

than by the physical capital k_t or output: a_t=q_t-k_{t+1}. So, it is convenient to transit from the state variable vectors (y,x) or (k,x) to pairs (q,x).

Next, as is usual in models of growth, we introduce effective state variables by normalizing q and x with respect to the corresponding (growing) aggregates:

$$kn_t = \frac{k_t}{\overline{K}_t},\ qn_t = \frac{q_t}{\overline{K}_t} = A_t {kn_t}^{\alpha} + (1-\delta)kn_t,\ xn_t = \frac{x_t}{\overline{X}_t}.$$

The law of motion for tokens in an agent's possession, (4), can be now rewritten as

$$xn_t = \frac{\overline{X}_{t-1}}{\overline{X}_t}\left(1 + R(\overline{X_{t-1}})\right)xn_{t-1} - \frac{S_t}{\overline{X}_t} + \frac{H_t}{p_t \overline{X}_t}.$$

But, combining (4), the expression for the total quantity of tokens in circulation, (5), and the market-clearing condition (9), one observes that

$$\overline{X}_t = \left(1 + R(\overline{X_{t-1}})\right)\overline{X}_{t-1},$$

so the law of motion for the effective individual token holdings is simply

$$xn_t = xn_{t-1} - \frac{S_t}{\overline{X}_t} + \frac{H_t}{p_t \overline{X}_t}.$$

Now we give the definitive form to the policy functions that will solve the agent's optimization problem. The purchased token quantity will be written as

$$H_t = q_t h(qn_t, xn_{t-1}) = \overline{K}_t qn_t h(qn_t, xn_{t-1}). \tag{10}$$

That is, the agent acquires the quantity of new tokens equal to the current gross disposable income times the normalized policy value h per unit of q, where h is assumed to be a function of current-period effective state variables. The latter are: xn_{t-1}, the effective, or aggregate growth-adjusted, number of tokens carried over from the last period, and qn_t, the effective gross disposable income generated by the last-period physical capital. Disparity of time superscripts between qn and xn is usual in models with investment, as the physical capital needed to produce today's output cannot be set instantaneously, i.e., it had to be determined a period earlier.

Recalling the non-negativity conditions on taxable income a_t-H_t=$q_t(1$-$h_t)$-k_{t+1}, we see that all h must lie between zero and one. Note that q_t-H_t=$q_t(1$-$h_t)$ is the part of the disposable income one decides not to hide in the crypto universe, i.e., the one to which the Leviathan's powers extend.[5]

The same non-negativity condition explains why it is useful to cast the investment policy function in the form

$$k_{t+1} = q_t\big(1 - h(qn_t, xn_{t-1})\big)v(qn_t, xn_{t-1}) \tag{11}$$

for some v, which is, like h, assumed to be a function of effective state variables and to lie between zero and one. Quantity v is the normalized future capital level per unit of "Leviathan-controlled" disposable income $q_t(1$-$h_t)$.

Expression (11) can be integrated over the agent population to render the following evolution law for the average per-capita physical capital:

$$\overline{K}_{t+1} = \overline{K}_t \int_\Omega qn(i)\left(1 - h\big(qn(i), xn(i)\big)\right) v\big(qn(i), xn(i)\big)\mu_t(di) = \overline{K}_t \widehat{K},$$

in which $\widehat{K}$, dependent on the time period only through the measure μ, expresses the aggregate capital growth rate between periods t and $t+1$. In particular, for ergodic measures, the capital growth rate is time-independent. The effective individual physical capital at time $t+1$is

$$kn_{t+1} = \frac{qn_t(1-h_t)v_t}{\widehat{K_t}}.$$

These transformations provide us with a convenient form of the law of motion for qn:

$$qn_{t+1} = A_{t+1}\left(\frac{qn_t(1-h(qn_t,xn_{t-1}))v(qn_t,xn_{t-1})}{\widehat{K_t}}\right)^\alpha + (1-\delta)\frac{qn_t(1-h(qn_t,xn_{t-1}))v(qn_t,xn_{t-1})}{\widehat{K_t}}. \tag{12}$$

Given (10) and (11), the expression (6) for current consumption can be restated as

$$c = n(q(1-h)(1-v)) + pL(l, S).$$

It remains to carry out the appropriate normalization of the token back-conversion variable S. It will be written as a multiple of the last-period individual token possession, x_{t-1}, and the normalized back-conversion quantity taken to be a function of effective state variables:

[5] Observe that here, as well as in the sequel whenever confusion is unlikely, we can replace the arguments of a function by a simple time subscript for simplicity.

$$S_t = x_{t-1} s(qn_t, xn_{t-1}). \quad (13)$$

To rule out token short-sales, we must impose the restriction $s_t \leq 1+R_t$ in all periods. Its formal justification will become clear shortly.

Let us rewrite the market-clearing condition (9) in the new notation. To do this, define the following *normalized price* as

$$pn_t = \frac{\int_\Omega qn(i) h(qn(i), xn(i)) \mu_t(di)}{\int_\Omega xn(i) s(qn(i), xn(i)) \mu_t(di)}. \quad (14)$$

Again, analogously to the aggregate capital growth rate $\widehat{K}$, this value depends on time only through the population asset distribution μ. In particular, pn is time-independent if μ is ergodic. One can interpret pn as the market-clearing crypto price in an economy in which the last-period aggregate capital and token quantity are both equal to one.

In terms of pn, (9) can be restated as

$$p_t = \frac{\bar{K}_t(1+R_t)}{\bar{X}_t} pn_t.$$

Combining this equality with (10), we can state the law of motion of xn as

$$xn_t = \left(1 - \frac{s_t}{1+R_t}\right) xn_{t-1} + \frac{qn_t h_t}{(1+R_t) pn_t}. \quad (15)$$

Together, (12) and (15) are the pair of state-transition equations that we are using in the equilibrium calculations. Denoting their right-hand sides as, respectively, F and G, we can symbolically shorten them to

$$qn_{t+1} = F\big(A_{t+1}, \widehat{K}_t; qn_t, xn_{t-1}\big), xn_t = G(\bar{X}_{t-1}, pn_t; qn_t, xn_{t-1}). \quad (16)$$

3.2 Individual optimality conditions

As is usual in optimal control models with a strictly concave objective function and smooth monotonic state-transition rules, first-order conditions (FOCs) are necessary for optimality in the interior of the feasible region for the control vector. Sufficiency of FOCs holds in that region as a consequence of the concavity of the utility and the production function. For a subset of initial conditions, corner solutions apply instead of FOCs (for one, two, or all three variables), which our numerical procedure is able to detect.

There are many equivalent ways to select the control variables. In our setting, it has proved convenient to make the following variables controls in each period: $b=H/p$ – the number of tokens purchased, $s=S/x$ – the proportion of currently held tokens to be converted into consumption units, k – the new physical capital level. For those, we compute partial derivatives of intertemporal utility U_t from section 2.3 to get

$$\frac{\partial U_t}{\partial b_t} = -p_t \sum_\lambda \pi_\lambda n'(a_t - p_t b_t) u'(c_t)$$

$$+\beta \sum\nolimits_{\lambda 1, \kappa 1} \pi_{\lambda 1} \vartheta_{\kappa 1} p_{t+1} s_{t+1} L_S(l(\lambda 1), x_t s_{t+1}) u'(c_{t+1}) \quad (17.1)$$

$$\frac{\partial U_t}{\partial s_t} = x_{t-1} p_t \sum_\lambda L_S(l(\lambda), x_{t-1} s_t) \pi_\lambda u'(c_t)$$

$$-x_{t-1}\beta \sum_{\lambda 1,\kappa 1} \pi_{\lambda 1} \vartheta_{\kappa 1} p_{t+1} s_{t+1} L_S(l(\lambda 1), x_t s_{t+1}) u'(c_{t+1}) \tag{17.2}$$

$$\frac{\partial U_t}{\partial k_{t+1}} = -\sum_{\lambda} \pi_\lambda n'(a_t - p_t b_t) u'(c_t)$$

$$+\beta \sum_{\lambda 1,\kappa 1} \pi_{\lambda 1} \vartheta_{\kappa 1} (A_{\kappa 1} \alpha (\bar{K}_{t+1})^{1-\alpha} (k_{t+1})^{\alpha-1} + 1 - \delta) n'(a_{t+1} - p_{t+1} b_{t+1}) u'(c_{t+1}). \tag{17.3}$$

In (17.2), L_S denotes the S-partial derivative of the conversion function L introduced in subsection 2.2. Since, if we fix the two other control variables, the current-period consumption gets into a one-to-one correspondence with the current-period token purchase, (17.1) can be considered the closest analogue of the standard Euler equation for this model. Further, (17.2) is an analogue of the conventional portfolio choice equation: it strikes the standard balance between the immediate benefit from selling tokens today and the future expected fall in benefit due to their reduced holdings next period. Finally, (17.3) generalizes the usual condition on the marginal product of capital. (In the absence of explicit risk-free borrowing or bonds in this model, the equation cannot be directly reduced to a more familiar intra-period equality.)

One obtains the FOCs for the agent's problem by equating the right-hand sides of (17) to zero. In our approach, it is a system of functional equations for the triplet of Markovian policy functions (h,s,v) of the two most recent state variable values qn and xn. In the last subsection, we already invoked the equalities

$$k_{t+1} == q_t(1\text{-}h_t)v_t, \quad a_t\text{-}p_t b_t = q_t(1\text{-}h_t)(1\text{-}v_t).$$

The first of them is needed to state the marginal product of capital equation (17.3) in terms of normalized states and decision variables. The second one is required to express the current-period consumption in these terms. The dependence on our choice variables, as was shown in the previous subsection, is the following:

$$c_t = n(q_t(1-h_t)(1-v_t)) + \frac{\bar{K}_t}{\bar{X}_{t-1}} pn_t L(l, \bar{X}_{t-1} xn_{t-1} s_t),$$

and the analogous equation should be applied to consumption in period t+1.

After transition to normalized variables, q_t becomes $\overline{K_t} qn_t$. Also, $s_{t+1} = s(qn_{t+1}, xn_t)$. In this equality, as well as everywhere else where they appear on the right-hand side of (17), the two state variables for date t+1 should be substituted for the right-hand sides of the state-transition equations (16). After that, the FOCs become a system of three functional equations for (h,s,v) dependent on the aggregate fundamentals pn_t, $\overline{K_t}$, $\bar{X}_{t-1}$, R_t as parameters. The optimal policy functions are those that either solve the FOCs while remaining inside the bounds: $0<h<1$, $0<s<1+R_t$, $0<v<1$, or are positioned at either end of the admissible interval, dependent on the corresponding derivative sign in (17). For instance, the optimal purchase rate of tokens is zero if the right-hand side of (17.1) is negative after substituting h=0, etc.

4 Equilibrium

In principle, an agent's optimal current-period decision in the environment defined above should have the form of a reaction function with the arguments being the agent's own (two-dimensional) asset endowment and the actions of all other agents. However, the agents here are assumed to be small and identical except for the aforementioned endowments. Agents with identical endowment pairs should also act identically. Accordingly, beside the own endowment variable, the reaction function effectively depends only on the asset endowment distribution in the agent population. Moreover, given the decisions, the asset distribution in one period uniquely determines this distribution in the subsequent one. One is left with the reaction function depending only on own endowment and the asset distribution "at the start of the history", i.e., in some initial period in the past. This initial distribution, known to everybody, will be assumed a primitive of the model. In short, our concept of equilibrium is a discrete-time analogue of the mean field game (MFG; see Lasry and Lions, 2007) equilibrium of continuous time dynamic games.

In view of the above, the equilibrium of the economy described in section 2 will be defined as the triplet of policy functions (h,s,v) such that

- the optimal controls in every period t for every agent with normalized endowment (qn_t,xn_{t-1}) are given by
  $$b_t=\overline{K_t}qn_t\,h(qn_t,xn_{t-1}),\ s_t=s(qn_t,xn_{t-1}),\ k_{t+1}=\overline{K_t}qn_t\,(1-h(qn_t,xn_{t-1}))v(qn_t,xn_{t-1})$$
- the aggregate levels of capital, tokens, and consumption are given by (2), (5), and (7), respectively
- token markets clear according to (9).

The asset distribution measure μ is fixed in the initial period. Its evolution in all subsequent periods is described by the following transition rule (an analogue of the Fokker-Planck equation of continuous time dynamic equilibria):

$$\mu_{t+1}([qn1,qn2]\times[xn1,xn2]) = \sum_{\kappa\in\mathrm{K}}\vartheta_\kappa\int\Theta_{[qn1,qn\ \ ]\times[xn\ \ ,xn2]}\left(F\left(A_\kappa,\widehat{K}_t;qn,xn\right),G(\bar{X}_{t-1},pn_t;qn,xn)\right)\mu_t(dqndxn) \tag{18}$$

for each rectangle $[qn1,qn2]\times[xn1,xn2]$. Symbol Θ_B in (18) stands for the indicator function of the set B in its subscript. When μ is ergodic, (18) becomes a functional equation for that measure.[6]

Observe that the optimality conditions for a given period t, based on (17), contain only one asset distribution measure, namely, μ_t. Therefore, one can compute optimal current-period policies for arbitrary μ_t if one is able to solve (17) numerically. Subsequently, one can adjust the solution iteratively to match aggregate equilibrium constraints (2), (5), (7), and (9). Although not guaranteeing an exact quantitative match with optimal policies for the subsequent period, this procedure is able to provide valuable qualitative insights into the optimal within-period behavior of a self-organizing agent population exposed, for example, to an exogenous asset endowment or productivity shock.

Our numerical solution procedure will then be based on first discretizing and subsequently interpolating policy functions (h,s,v) on an admissible (q,x)-domain and solving the FOCs implied by (17). In the next subsection, we show some computation results for an arbitrarily chosen asset distribution μ. After that, we examine the results for ergodic μ in subsection 4.2.

4.1 Non-ergodic initial distribution of asset holdings

3D graphs of optimal policy functions under lognormal physical wealth and token endowment distribution (mutually independent) are shown in Fig. 3.

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[6] Note that the current discussion refers to the case of idiosyncratic uncertainty. Analysis of systemic shocks is possible along the same formal lines, since, in this model, such a shock would be formally manifested as a jump-change in one or more of the wealth distribution parameters (for example, an aggregate productivity shock would mean a shift in the physical wealth mass).

The least surprising is the behavior of the new token purchase rate (in relation to current income) featured in panel (a): the richer one is, the more tokens one purchases. There is also a clear feedback loop in the demand for tokens: it is higher, the more tokens the agent already holds. In the extreme, when the agent does not have any tokens initially, he does not buy any unless he is extremely rich (in our example, it is the richest 1 per cent).

The token conversion policy, shown in panel (b), is less straightforward. Individuals with below-average physical possessions get rid of all their tokens at the end of the period, provided they had a low position in them initially. However, even agents with below-average physical wealth, if they start the period with an above-average amount of tokens, have a different attitude; they sell back between one half and two thirds of their token position. Agents in the upper range of the physical wealth spectrum are the best predictable: they sell relatively less tokens when they start with less, and relatively more when they start with more. Altogether, the surface of this policy function graph exhibits many reversals in all directions and does not allow for a unified intuitive interpretation in its entire range.

Of special interest is the physical investment policy (panel (c)), since it allows for a direct comparison with economies without crypto. It turns out that a poor agent invests almost identically regardless of the presence of tokens (indeed, recall that such agents either ignore the crypto option completely or sell the few tokens they have purchased, within the same period). The difference begins to grow with wealth. Whereas in the cryptoless economy, the investment rate bottoms out with growing wealth (and even starts to rise in the range of very high wealth levels), an agent with zero or low initial token holdings has the investment rate as a strictly decreasing function of initial wealth everywhere. This happens because, as the agent becomes wealthier, crypto becomes increasingly relevant for him, so it pays to forego a part of future physical investment income in order to consume more from token turnover (tokens allow the agent to escape growing societal complexity). Another noteworthy effect is related to agents with a substantial initial token position. As this position gets bigger, the range of wealth levels for which it is optimal to choose the maximal admissible physical investment rate (v=1), gets wider. In that range, agents split current earnings between new physical investment and token purchase, spending nothing on consumption. All consumption is payed for out of proceeds from pre-existing token back-conversion. To be willing to partially consume out of conventional income, a crypto-affluent agent must also be very affluent in conventional terms.

The qualitative properties of the optimal policies described above are not specific to the particular choice of initial asset distribution. The distribution becomes quantitatively relevant for aggregate fundamentals, including the token clearing price and the growth rate.

4.2 Ergodic asset distribution

To derive the equilibrium under the ergodic asset measure, one needs the measure and the policies that simultaneously solve (17) and (18). Thanks to the low sensitivity of the optimal policy shapes to the exact choice of measure, one can iterate between the two equations to achieve convergence.

Naturally, since numerical solutions require discretization, we do not solve (18) as it is stated in integral form, but use summation over a two-dimensional grid. Then, instead of a continuous bivariate density function, we look for a vector md of masses of individual cells (rectangles) in the chosen grid that satisfies the appropriately discretized version of (18). The right-hand side of this discretized version happens to be a linear operator, i.e., a very high-dimensional but sparse matrix H. For our computational purposes it is important that H maps the unit simplex into itself, i.e., every vector md with non-negative components summing up to unity is transformed into a vector with the same properties. In other words, H preserves the unit simplex of discrete probability measures. For matrix

H to satisfy this property it is necessary and sufficient for its (non-negative) elements in each column to sum up to one. Unless one considers degenerate policy functions, the property is satisfied due to the regularity of the state transition map $(qn, xn) \mapsto \left(F\left(A, \hat{K}; qn, xn\right), G(\bar{X}, pn; qn, xn)\right)$ for every $\left(A, \hat{K}, \bar{X}, pn\right)$.

Since the unit simplex is a compact subset of the Euclidean space, H must possess at least one fixed point. It is, actually, unique, since multiplicity would violate the monotonicity properties of functions F and G. This unique fixed point $\widehat{md}$ is the discrete approximation of the ergodic measure we are looking for. A close inspection of the structure of H suggests an educated guess that the support of $\widehat{md}$ must be located around the convex hull of the fixed points of the aforementioned state transition map for individual realizations of the TFP shock A. This guess allows one to select a suitable zero iteration for recursive computation of $\widehat{md}$. The iterative procedure happens to converge quite quickly. Then, alternating between solutions to discrete approximations of (17) and (18), one arrives at the numerical approximation for optimal policies in the ergodic equilibrium.

Of the wide range of results one can obtain from inspecting the ergodic numerical solution, we select the observations referring to the comparison of ergodic states of economies with and without tokens. To begin with, one finds that investment rates are much less sensitive to the current physical wealth level when tokens are absent (Fig. 4). Qualitatively, this finding holds for other than ergodic asset distributions as well. However, in the ergodic case, the behavior of agents with a zero initial token position differs more visibly from the tokenless case than in the lognormal distribution case studied in subsection 4.1. Not surprisingly, the agents in the ergodic state have had sufficient time to internalize the potential of token utilization even if they currently happen to have zero crypto holdings. Another observation is that the aggregate capital growth rate $\hat{K}$ in the ergodic case is higher when tokens are present: agents are, on average, wealthier also in physical capital terms (Fig. 5).

We also collect results on the behavior of some key aggregate fundamentals in the ergodic equilibrium in Table 2 and, where relevant, compare them with the corresponding fundamentals in the tokenless economy under ergodicity of physical wealth distribution. What we observe immediately is that the market for tokens always has both sellers and buyers (it does not dry up), and the aggregate number of tokens grows at a positive rate.

Another noteworthy finding is that the presence of tokens contributes to higher physical capital and aggregate consumption growth rates. This happens in spite of the fact that very rich persons invest less in physical capital (in favor of crypto) than in the tokenless economy. However, the aggregate physical growth rate is enhanced by the token presence because of the "middle class". That is, more people of medium wealth can afford substantial (relative to wealth) physical investment when their consumption can be co-financed by crypto sales. This happens because, by channeling their expenditure through the crypto ecosystem, they are able to avoid some of the Leviathanian waste. Of course, in the present model the Leviathan is only present on the consumption side. If the model allowed social complexity costs to affect the production side as well, the result may have been different.

As regards the properties of the ergodic distribution itself, the most important, in our view, is the non-negligible positive correlation between physical and crypto wealth (the last row of Table 2). That is, even if the economy starts with independently distributed physical capital and token endowments, it tends to converge to a state in which conventional and crypto wealth are partially aligned. Another salient feature of the ergodic distribution is a much higher variance of token possessions compared to physical capital (not shown in the table). However, in this case, the effect may be partially due to the discretization procedure, which, combined with the restriction of permanent token market clearing, generates some additional "raggedness" in the distribution of tokens across the agent population. Therefore, we refrain from drawing far-reaching conclusions from this feature.

5 Conclusion

We have defined a dynamic model of a growing economy with stochastic productivity realizations and increasing complexity consequences of wealth accumulation, and allowed the existence of alternative (crypto) assets in this economy, the conversion of which into real assets lacks legal protection and is subject to random losses. For this world with "mainstream" and "alternative" wealth universes existing in parallel, we have developed a method of equilibrium derivation that, under many circumstances, demonstrates the existence of a multitude of agents who hold assets in both. There is also a "crypto non-adoption" subspace of (small) wealth value pairs; agents with endowments in this subspace do not use crypto assets.

We define and provide a procedure for approximate calculation of the ergodic asset distribution across the agent population, i.e., we show the possibility of a long-term stable configuration of portfolios invariant under the dynamic rules of the economy constructed. This ergodic distribution, among other things, characterizes the long-term dynamic behavior of the crypto asset volume, the growth rate of which is a part of the balanced growth equilibrium. Altogether, when the economy starts with some degree of wealth heterogeneity across otherwise identical agents, there exist both short-term and long-term conditions for coexistence of conventional and crypto asset markets, with the latter being predominantly used by successful participants in the former. Idiosyncratic shocks that give rise to agent heterogeneity in the analyzed setting are not to be confused with the potential aggregate shocks (including, among other things, shocks to the same variables) able to radically change the adoption pattern, and aggregate implications of, crypto assets. This remains a matter for further research.

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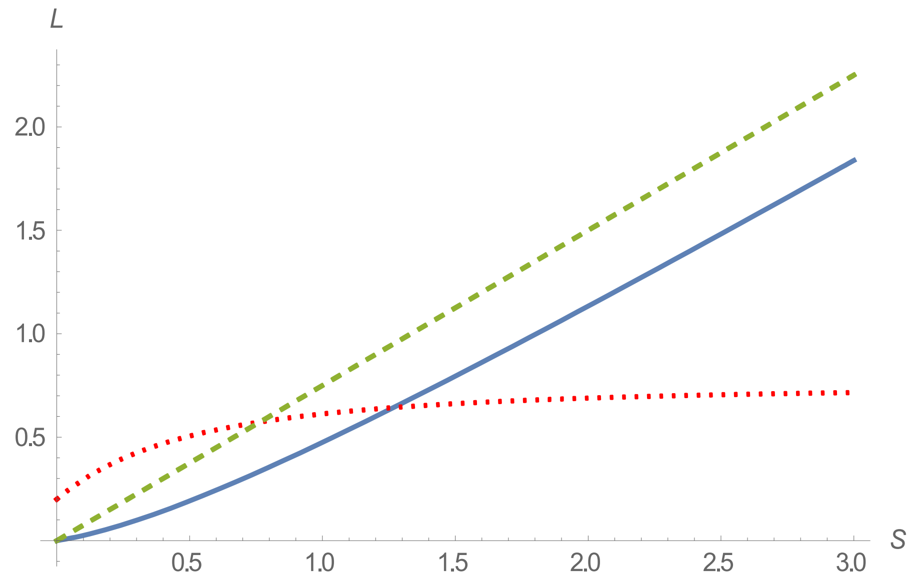
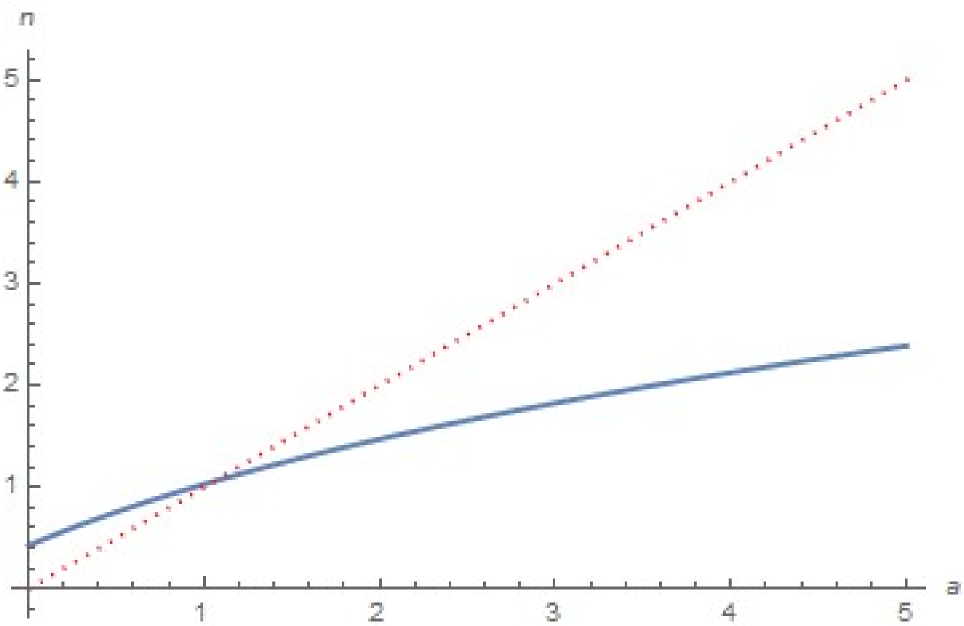
Fig. 1 Revenue from converting token amount *S* back to the consumption numéraire



Note: shown are the conversion revenue $L(l,S)$ for l=0.75 (solid line), its asymptote with slope l (dashed line), and the marginal revenue (dotted line).

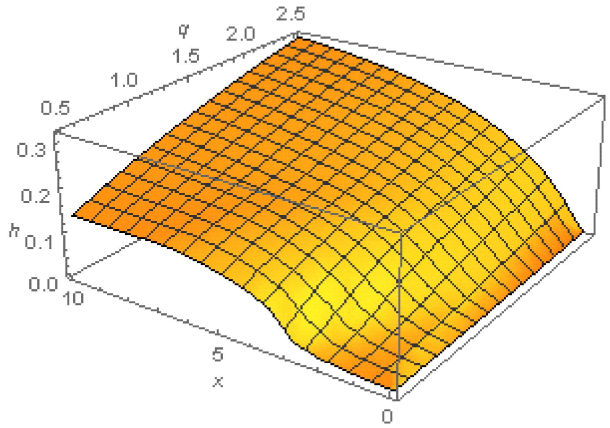
Fig. 2 Effective absorption curve as a function of disposable income in an economy with increasing social complexity



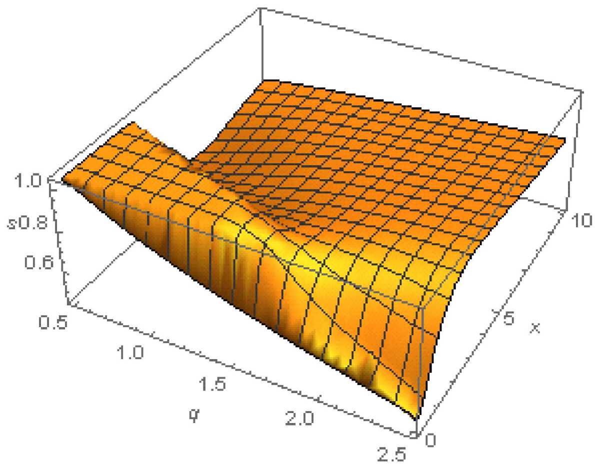
Note: the dotted line is the diagonal.

Fig. 3 Optimal policies under mutually independent lognormally distributed endowments of physical wealth and crypto holdings

(a) New token purchase



(b) Token back-conversion



(c) Physical investment

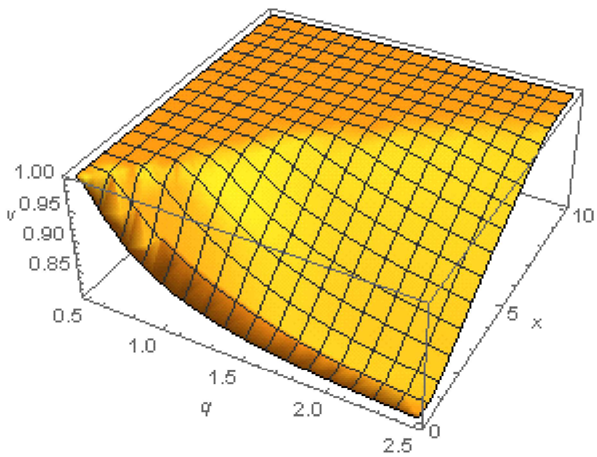
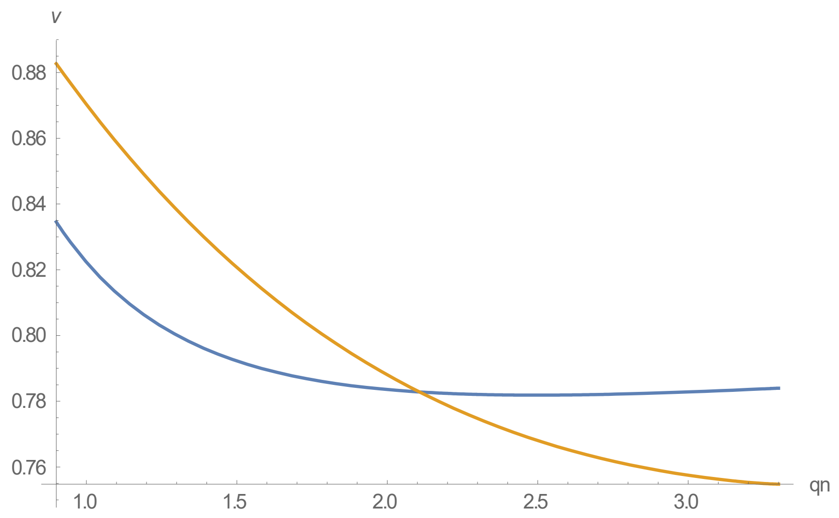
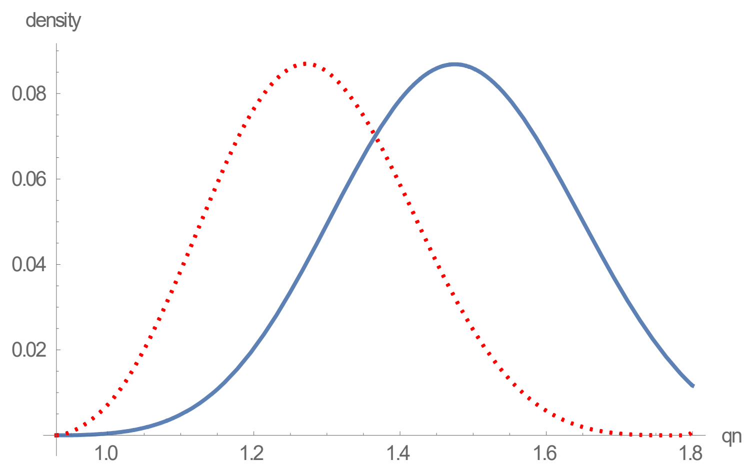


Fig. 4 Investment policy in an economy with and without tokens



Note: the orange line shows the investment rate of an agent with zero initial token holdings; the blue line is the investment rate in the tokenless economy

Fig. 5 Marginal physical wealth density of the ergodic distribution of asset pairs, compared to the physical wealth density in the tokenless economy



Note: the solid line is the marginal physical wealth density in the economy with crypto; the dotted line is the physical wealth density in the tokenless economy

Table 1 Parameterization

Variable/Function	Description	Value/Functional form
α	Capital share	0.33
β	Time preference rate	0.97
δ	Capital depreciation rate	0.03
ρ	Share of current consumption in the consumption index	0.9
σ	Consumption substitution elasticity	3
$n(a)$	Ex-complexity cost absorption as a function of disposable income	$\frac{1}{\tau}((a_0 + a)^{\tau} - a_1)$
$L(l,S)$	Crypto conversion function	$\frac{l_0 S + l S^2}{1 + S}$
$R(X)$	Rate of return on crypto as a function of the aggregate number of tokens	$\gamma_0 \left(1 + \frac{\gamma - \gamma_0}{\gamma_0} \frac{X}{1 + X}\right)$

Table 2 Aggregate economic indicators in the ergodic equilibrium

Fundamental description	Notation	Economy with crypto assets	Economy without crypto assets
Physical capital growth rate	$\hat{K}$	1.094	1.036
Normalized token price	pn	0.071	-
Consumption growth rate	$\hat{C}$	1.081	1.005
Token growth rate	$R(\bar{X})$	0.032	-
Correlation between physical and crypto wealth	$Corr[qn, xn]$	0.142	-